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Atmosphere dynamics

Chapter-1

EQUATION OF MOTION

1.1. Inertial or absolute coordinate system:

Inertial coordinate system is one, which holds good the Newton's law of acceleration. The Newton's law states that the applied force is equal to the product of the mass and the acceleration produced. A Cartesian coordinate system, which is fixed with respect to the fixed stars, is very nearly an inertial frame. However, the one that moves with some acceleration is called the 'non inertial coordinate system' in the space.

1.2. Rotating Coordinate System:

Newton's law depends entirely upon the nature of coordinate system relative to the surface on which it is acting. So on the surface of the earth, this coordinate system also rotates with the earth about its axis of rotation. As a result the departures of acceleration under such a rotating coordinate system have to be worked out. The coordinates on the rotating system of earth taken are positive 'x' as latitude towards east, positive 'y' along the meridian towards the North Pole and 'z' as the height into the atmosphere as shown in Fig.1.

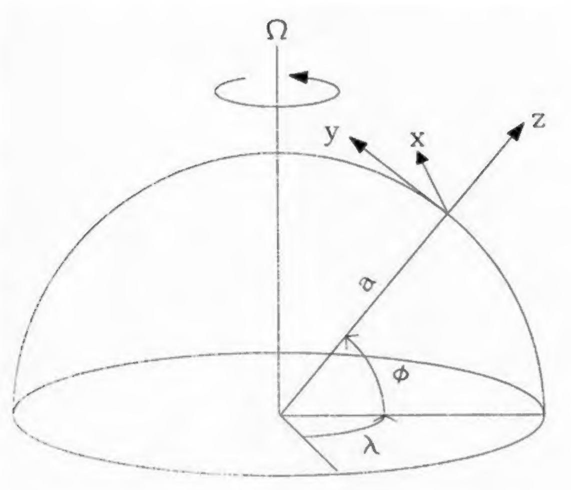


Fig.1. Coordinates on the spherical earth

1.3. Equations in a rotating Coordinate System:

Let us consider a coordinate system with the coordinates x',y' and z' as shown in Fig.2. Let the coordinate system rotated by an angle Ωt and the new coordinates are x,y and z respectively. From the rotational coordinate system we can write

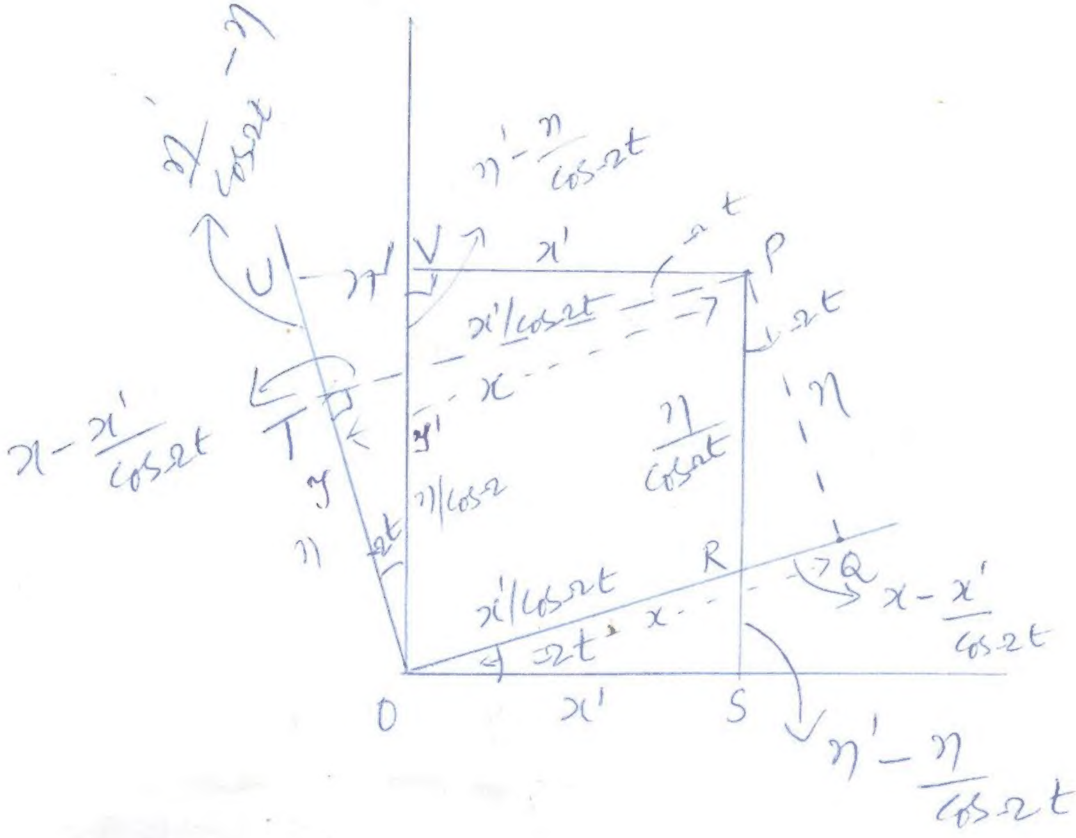


Fig.2. Components on the rotating coordinate system

From triangle ORS, OR = $\dfrac{X'}{Cos\Omega t}$,

From triangle PQR, QR = OQ-OR = $X - \dfrac{X'}{Cos\Omega t}$; PR = $\dfrac{\eta}{\cos\Omega t}$

From triangle PQR, $Sin\Omega t = \dfrac{QR}{PR} = \dfrac{X - \dfrac{X'}{Cos\Omega t}}{\dfrac{\eta}{\cos\Omega t}} \quad \Rightarrow X' = XCos\Omega t - \eta Sin\Omega t$

Similarly from triangle PTU, PT = X, From triangle OUV, TU = $\dfrac{\eta'}{\cos\Omega t} - \eta \left(\because OT = \eta, OU = \dfrac{\eta'}{\cos\Omega t} \right)$

From triangle PTU, $Tan\Omega t = \dfrac{TU}{PT} = \dfrac{\dfrac{\eta'}{\cos\Omega t} - \eta}{X} = \dfrac{Sin\Omega t}{Cos\Omega t}$

$\therefore \eta' = X \sin \Omega t + \eta \cos \Omega t$

It may be noted that the letter 'Y' may be read in place of η then the we get the following equations.

$$X' = X \cos \Omega t - Y \sin \Omega t$$
$$Y' = X \sin \Omega t + Y \cos \Omega t \qquad (1.1)$$
$$Z' = Z$$

Similarly in place of X, Y, and Z if forces F_x, F_y, and F_z are considered we can write

$$F_x' = F_x \cos \Omega t - F_y \sin\Omega t$$
$$F_y' = F_x \sin \Omega t + F_y \cos \Omega t \qquad (1.2)$$
$$F_z' = F_z$$

If we differentiate equation 1.1 twice with respect to time we get

$$\ddot{X}' = Cos\Omega t(X\ \ddot{X} - 2y\Omega - X\Omega^2 \text{ - } \sin\Omega t(\dot{y} + 2\dot{x}\ \Omega - y\Omega^2)\ldots\ldots\ldots\ldots\ldots(1.3)$$

According to Newton's law, $a = F/m = \ddot{X}$ ……………………………..(1.4)

So equation 1.2 and 1.4 combining we write

$$F'_x/m = \left[F_x/m\right]\cos\Omega t - \left[F_y/m\right]\sin\Omega t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1.5$$

since $\ddot{X}' = F'_x/m$ Equations 1.3 and 1.5 can be equated. After equating and comparing the coefficients of Cos Ωt and Sin Ωt we get as below:

$$\ddot{X} - 2\dot{Y}X - X\Omega^2 = F_x/m \quad \text{or} \quad \ddot{X} = X\Omega^2 + 2\dot{Y}X + F_x/m$$
$$\ddot{Y} + 2\dot{X}\Omega - Y\Omega^2 = F_y/m \qquad \ddot{Y} = Y\Omega^2 - 2\dot{X}\Omega + F_y/m \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.6)$$

Similarly can be written $\ddot{Z} - \frac{F_z}{m} = 0$

Equations (1.6) are Newton's second law transformed to a rotating coordinate system.

The first of the terms on the right hand side ($X\Omega^2$, $Y\Omega^2$) denote the centrifugal accelerations pointing radially outward. The second terms on the right hand side ($2\dot{Y}X, 2\dot{X}\Omega$) represent accelerations owing to the combined effects of rotation of the coordinate axes and motion of the particle, relative to the rotating coordinate system. These are called Coriolis accelerations. The action of this Coriolis acceleration is:

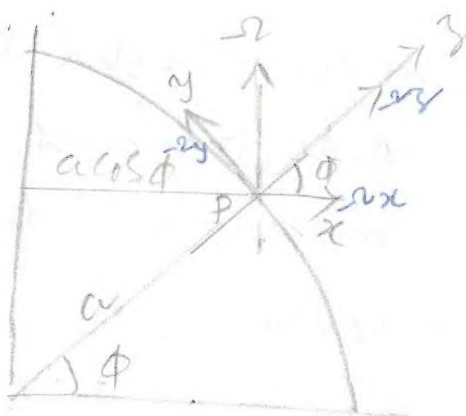
1. It acts only perpendicular to the plane of the moving particle.

2. It acts right hand side in the N.H and left hand side in the S.H of the moving particle.
3. It comes into being only for a moving particle but not for a fixed particle.
4. It does not change the magnitude of the velocity but changes only the direction of the moving particle as it acts perpendicular to the plane of moving particle.

Thus the centrifugal and Coriolis accelerations are extra terms obtained due to non inertial (non Newtonian) frame of reference.

1.4. Spherical coordinate system:

We shall now apply these results on a rotating spherical earth, in which x is tangent eastward, y meridionally northward and z radially upward into the atmosphere as shown in Fig.3. Let 'P' be a point on the earth's surface which is rotating with the angular velocity Ω, at a distance of 'acosφ' from the axis of rotation where 'a' is the radius of the earth and 'φ' is the latitude. From fig.3, $\Omega_x = \Omega \cos 90 = 0$, $\Omega_y = \Omega \sin(90-\varphi) = \Omega \cos \varphi$, $\Omega_z = \Omega \sin\varphi$.



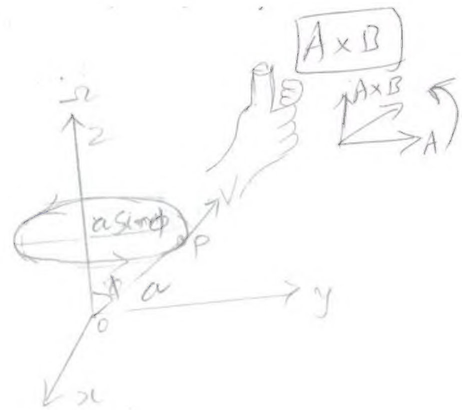


Fig.3 a &b

If the velocity of the particle is V= iu +jv+kw then the Coriolis acceleration is curl of the vector as shown in fig.3b. The direction of the vector Ω is such that when the fingers of the right hand bend(curl) about the axis of rotation in the same sense as the motion, the thumb points in the direction of Ω.

In fig.3b let 'a' is the position vector of the moving particle'P' as measured from the origin of coordinates (centre of the earth). So the vector Ω×a has the magnitude 'Ω a sinφ', which is just the linear speed of the particle (V). Therefore V = Ω×a. Similarly if the magnitude of velocity (V) is considered as the position vector, then the acceleration is Ω×V. Thus we got the Coriolis accelerations in equation 1.3 as $-2\dot{X}\Omega and -2\dot{Y}\Omega$, where $\dot{X} and \dot{Y}$ are the velocities $(\frac{dx}{dt}, \frac{dy}{dt})$. Which means the Coriolis accelerations on the surface of the earth can be written as -2Ω×V.

$$C = -2\Omega\times V = -2\Omega\begin{vmatrix} i & j & k \\ 0 & \cos\phi & \sin\phi \\ u & v & w \end{vmatrix} = -2\Omega\begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

where u,v,w are the x,y,z components of the velocity. Expanding the determinant and separating the different components we get,

$$C_x = -2\Omega\, w \cos\varphi + 2\Omega v \sin\varphi$$
$$C_y = -2\Omega u \sin\varphi \qquad \text{……………………(1.7)}$$
$$C_z = 2\Omega\, u \cos\varphi$$

The Newton's second law of motion on a rotating coordinate system becomes from equation 1.6 as

$$\ddot{X} - \frac{F_x}{m} = 2\Omega v Sin\phi - 2\Omega w Cos\phi$$
$$\ddot{Y} - \frac{F_y}{m} = -2\Omega u Sin\phi \qquad \text{…………………..(1.8)}$$
$$\ddot{Z} - \frac{F_z}{m} = 2\Omega u Cos\phi$$

1.5. Other forces or accelerations:

1.5.1. Gravitation and Gravity:

The term gravitation solely means the attractive force between bodies which is described by Newton's law of gravitation. This force of attraction is proportional to the product of the masses and inversely related to the square of the distance between them.

$$g_a = -(GM/r^2)r$$

Where G= 6.658×10^{-8}, M = 5.988×10^{27} grams is the mass of the earth, r = the position vector of the unit mass of the air molecule measured from the centre of the earth.

As the particle on the surface of the earth is rotating along with the earth, it has centrifugal acceleration and the net effect appears as acceleration due to gravity as shown in Fig.4.

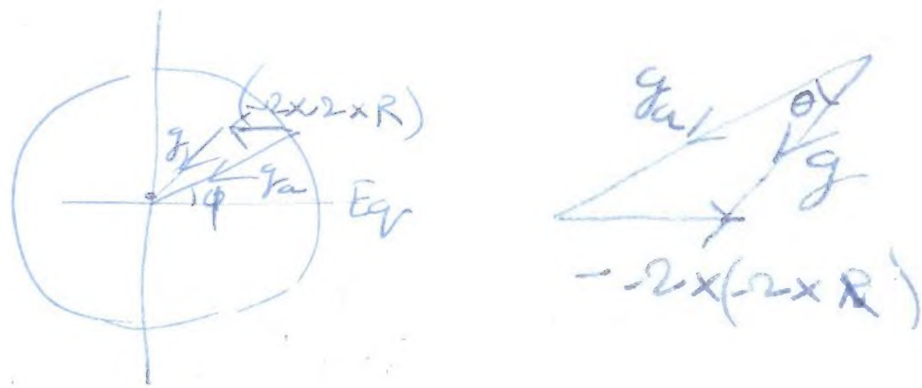


Fig.4 Gravitational and centrifugal accelerations on the earth

$$g = g_a - \Omega\times(\Omega\times r)$$

which is equal to acceleration due to gravity = gravitation – centrifugal acceleration.

At the surface of the earth acceleration due to gravity(g) depends on the latitude. At poles it is maximum and minimum at the equator.

1.6. The Pressure gradient force

Consider a rectangular volume having the sides $\delta_x, \delta_y and \delta_z$ fixed in a coordinate system relative to the solid earth as shown in Fig.5. Then the force in the x-direction due to the atmospheric pressure is, $p\delta_y\delta_z$ where 'p' is the atmospheric pressure and $\delta_y\delta_z$ is the area of the face it is acting.

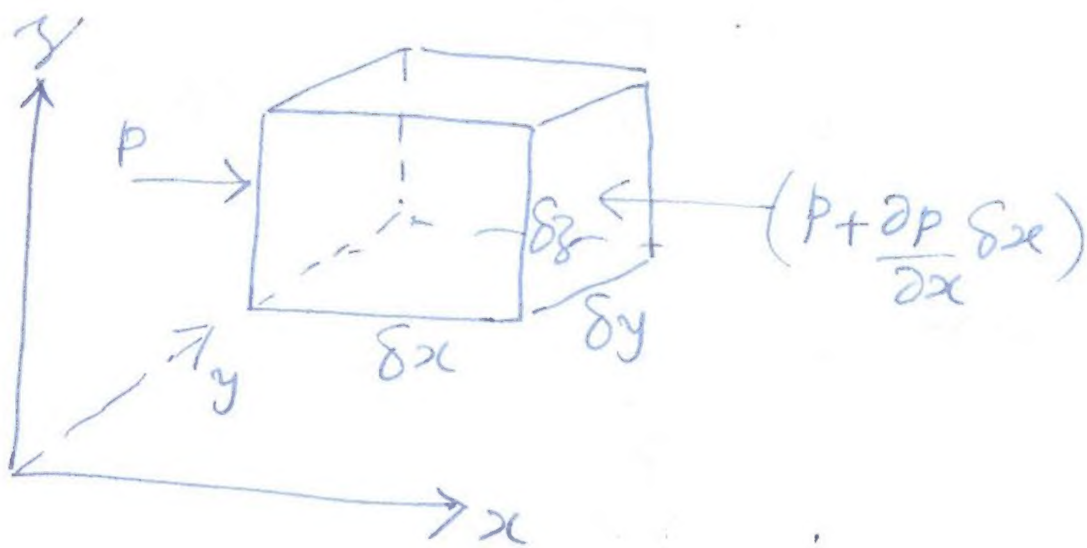


Fig.5. Pressure force acting on a rectangular volume element

Similarly the pressure force acting on the opposite face would be - $\left(p+\frac{\partial P}{\partial x}\delta x\right)\delta y\delta z$

The net force acting in the x direction (i) would be

$(ip\delta y\delta z)+\left\{-i(p+\frac{\partial p}{\partial x}\delta x)\delta y\delta z\right\}=-i\frac{\partial p}{\partial x}\delta x\delta y\delta z$, pressure force per unit volume= -$i\partial p/\partial x$ because the volume of the cube is $\delta x\delta y\delta z$.

Similarly considering in other directions the total pressure force would be

$$-\left\{i\frac{\partial p}{\partial x}+j\frac{\partial p}{\partial y}+k\frac{\partial p}{\partial z}\right\}$$

The force per unit mass = $-\frac{1}{\rho}\left\{i\frac{\partial p}{\partial x}+j\frac{\partial p}{\partial y}+k\frac{\partial p}{\partial z}\right\}$ = $-\alpha\nabla p$ …………(1.9)

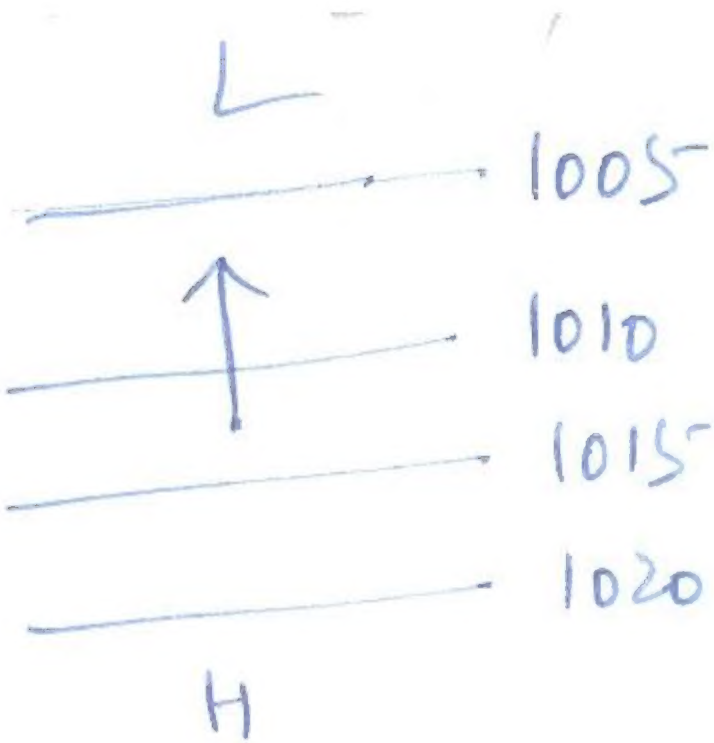


Fig.6. As the pressure increases top to bottom, pressure force acts in the opposite direction

The minus sign indicates that as the pressure increases in one direction, the pressure force acts in opposite direction as shown in Fig.6. Because of the presence of gradient (del,∇) operator, it is called as the pressure gradient force.

1.7. FINAL EQUATION OF MOTION:

Finally including all the forces, the vector equation of motion can be written as (from equations 1.7):

Table 2.11

Total acceleration	Coriolis acceleratio n	Pressure gradient	Acceleration due to gravity	Friction
$\ddot{X} = \frac{d^2x}{dt^2} = \frac{du}{dt}$	f.v- 2Ωwcosφ	$-\alpha\frac{\partial p}{\partial x}$	0	F_x/m
$\ddot{Y} = \frac{d^2y}{dt^2} = \frac{dv}{dt}$	-f.u	$-\alpha\frac{\partial p}{\partial y}$	0	F_y/m
$\ddot{Z} = \frac{d^2z}{dt^2} = \frac{dw}{dt}$	+2Ωucosφ	$-\alpha\frac{\partial p}{\partial z}$	-g	F_z/m

Then we can write the Navier- stokes equations as:

$$
\begin{aligned}
\frac{Du}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + (2\Omega v \sin\phi - 2\Omega w \cos\phi) \\
\frac{Dv}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - (2\Omega u \sin\phi) \qquad (4.19) \\
\frac{Dw}{Dt} &= -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + (2\Omega u \cos\phi)
\end{aligned}
$$

acceleration of air parcel | gravity | pressure gradient force | viscous force | Coriolis force

(Or)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$+(2\Omega v\sin\phi-2\Omega w\cos\phi)$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-(2\Omega u\sin\phi)$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-g-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+(2\Omega u\cos\phi)$$

This set of equations is usually called the Navier–Stokes equations for the conservation of momentum. The equations are named after Claude Louis Navier (1785–1836) and George Gabriel Stokes (1819–1903), who both contributed to the development of the equations. Navier developed the form of these equations for an *incompressible* (that is, constant density) fluid in 1821. In 1822, he published a further refinement

1.8: SCALING THE EQUATION OF MOTION:

Scaling is useful to know which term weighs more than the other terms in the equation of motion and so the terms that weigh more can be retained and the others can be neglected or ignored for different calculations. The procedure is to take approximate rounded off values (order of magnitude) of each term and substitute as per the dimensional value of the each term as given in the table below:

1.8.1. Scaling the different terms of the equation of motion in the case of tropical weather systems:

Item	Order of magnitude
1. horizontal velocity u ,v	10 m/s = 10^3 cm/s
2. vertical velocity	1 cm/s
3.horizontal length of a synoptic system,Δx,Δy	1000 km, =10^8 cm
Vertical depth of the system, Δz	10 km, = 10^6 cm
5. pressure fluctuation, Δp	10 mb = 10^4 dynes/cm^2
6.time dimension	10^5 seconds

1. Acceleration: du/dt = $10^3/10^5 = 10^{-2}$
2. Coriolis terms:
a) $2\Omega v \sin\varphi = 2\times7.292\times10^{-5}\times\text{Sin}90^o\times v = 10\times10^{-5}\times1\times10^3 = 10^{-1}$. So 2Ω v sinφ is valid.
b) $2\Omega w \cos\varphi = 2\times7.292\times10^{-5}\times\text{Cos}(0)\times w = 10\times10^{-5}\times1\times1 = 10^{-4}$.

Which means $2\Omega w \text{Cos}\varphi$ is negligibly smaller than $2\Omega v\text{Sin}\varphi$. So $2\Omega w\text{Cos}\varphi$ can be neglected.

c) $2\Omega u\text{Sin}\varphi = 2\times7.292\times10^{-5}\times\text{Sin}90\times u = 10\times10^{-5}\times1\times10^{3} = 10^{-1}$, valid.

d) $2\Omega u \text{ Cos } \varphi = 2\times 7.292\times10^{-5}\times\text{Sin}90\times u = 10\times10^{-5}\times1\times10^{3} = 10^{-1}$

3. Gravity term: $g = 9.8 = 10 m/s^2 = 10^3$ cm.

4. Pressure gradient terms:

$\alpha\partial p/\partial x$, $\alpha\partial p/\partial y = (1/\rho)\partial p/\partial n = 10^4/(10^{-3}\times10^8) = 10^{-1}$

$\alpha\partial p/\partial z = (1/\rho)\partial p/\partial z = 10^4/(10^{-3}\times10^6) = 10^1$

i) Scaling x equation: (frictionless flow):

$\frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + 2\Omega v\sin\Phi - 2\Omega w\cos\Phi = 10^{-2} = 10^{-1} + 10^{-1} - 10^{-4}$ The last term is negligibly small compared to the other two terms in the right hand side and so can be neglected.

$$\therefore \frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + 2\Omega v\sin\Phi$$

ii) Scaling y-equation:

$\frac{dv}{dt} = -\alpha\frac{\partial p}{\partial y} - 2\Omega u Sin\Phi \rightarrow 10^{-2} = 10^{-1} - 10^{-1}$, both terms in the right hand side are equal and so all are important.

iii) Scaling z- equation:

$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} + 2\Omega u\cos\Phi - g \rightarrow 10^{-5} = 10^1 + 10^{-1} - 10^3$, when compared to pressure gradient and gravity, $2\Omega u \cos\varphi$ is negligibly small.

$$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} - g$$

Thus final scaled equation of motion is

$$\therefore \frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + 2\Omega v\sin\Phi$$

$$\frac{dv}{dt} = -\alpha\frac{\partial p}{\partial y} - 2\Omega u Sin\Phi$$

$$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} - g$$

Chapter-2

2. HYDROSTATIC EQUATION & GEOPOTENTIAL:

The third equation of motion reduces to hydrostatic equation as dw/dt and $2\Omega u \cos\varphi$ are neglected.

Therefore, $-\alpha\partial p/\partial z - g = 0$,

$$\frac{\partial p}{\partial z} = \rho g \Leftrightarrow \alpha dp = gdz \Rightarrow \int \alpha dp = \int gdz \Rightarrow \int d\Phi = \int gdz$$

This φ is called the geopotential and has the dimensions L^2T^{-2}. The unit of geopotential is dynamic meter which is numerically 2% smaller than the geometric meter because while the work needed to lift a unit mass through one geometric metre is about 0.98 dynamic meters, which is close to 1.0 geopotential meter. The value gdz is the potential energy between the thicknesses of two levels. Constant geopotential lines are called equigeopotential lines which are also level surfaces. To move along an equigeopotential line no extra energy is required as the acceleration due to gravity is constant on the equipotential lines. The equigeopotential surfaces on the earth are oblate spheroidal like that of the earth as equatorial radius is larger than the polar radius.

2.1. THE EQUATION OF CONTINUITY:

An important restriction upon the velocity of a fluid may be obtained by kinematic methods applied to the law of conservation of mass.

Consider an infinitesimal rectangular volume fixed in space with sides of lengths δx, δy,δz in a moving fluid having velocities u,v,w respectively as shown in Fig.7. Consider first the flow parallel to the x-axis. As 'u' is x- component of velocity and ρ is density of the fluid the mass flow through the left face is ρu.

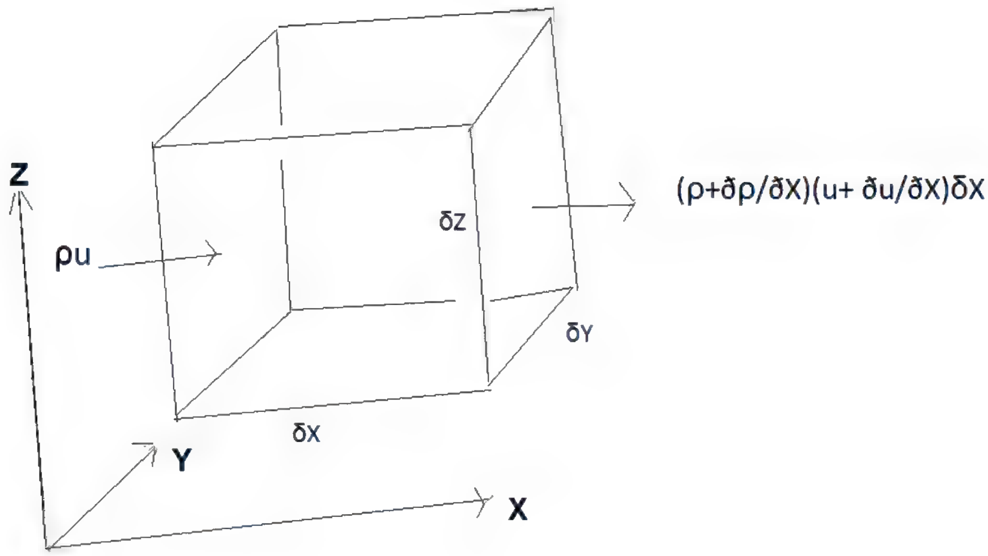


Fig.7.

The mass flow into the volume per unit time is $(\rho\ u)\delta y\delta z$

The mass flow out of the volume per unit time on the right face is

$$(\rho + \frac{\partial \rho}{\partial x}\delta x)(u + \frac{\partial u}{\partial x}\delta x)\delta y\delta z$$

So the net flow out of the volume in the x-direction is

$$(\rho u\delta y\delta z) - \left[(\rho + \frac{\partial \rho}{\partial x}\delta x)(u + \frac{\partial u}{\partial x}\delta x)\right]\delta y\delta z$$

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}\delta x\right]$$

As the volume of the rectangular element considered is infinitesimal, δx is very small and so higher order term (second term) is negligible.

Therefore, the net flow in the x-direction is $= -\left[\frac{\partial}{\partial x}(\rho u)\right]\delta x\delta y\delta z$

Taking the mass flow in all the three component directions, the total flow out would be $-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\delta y\delta z$

where 'v' and 'w' are the velocity components in the y and z directions respectively.

We shall compute the rate at which mass enters the box as a result of fluid motion and, since mass must be conserved, we shall equate this to the rate of change with time of the mass contained within the cube.

In other words, if mass is neither created nor destroyed and if the sum is not zero, it must be balanced by a change of density (rate of change of density $\frac{\partial \rho}{\partial t}$)

$$\therefore \left(\frac{\partial \rho}{\partial T}\right)\delta X \delta Y \delta Z = -\left[\frac{\partial}{\partial X}(\rho U) + \frac{\partial}{\partial y}(\rho V) + \frac{\partial}{\partial z}(\rho W)\right]\delta x \delta y \delta z$$

$$-\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) + \frac{\partial}{\partial z}(\rho W) = \nabla . \rho \bar{V} \ldots\ldots\ldots\ldots(2.1)$$

$$\frac{1}{\rho}\frac{d\rho}{dt} + \left[\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z}\right] = 0 \ldots\ldots(\ 2.2)$$

This is called the equation of continuity. The first term is the fractional rate of change of density and the second term is the fractional rate of change of volume.

If the fluid is incompressible, the density is uniform (constant), then $\frac{d\rho}{dt} = 0$

$\therefore \frac{1}{\rho}\frac{d\rho}{dt} = 0$ Then the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla . \bar{V} = 0 \ldots\ldots\ldots\ (2.3)$$

$where\, \bar{V} = iu + jv + kw$

Equation (2.3) is frequently considered convenient as it is nothing but horizontal three dimensional divergence.

Chapter -3
Circulation and vorticity

3.1. CIRCULATION:

Let the fluid is flowing along a path ‘ s’, a portion of it is say ‘ds’, with the components dx and dy, l = cos β = dx/ds, cos θ = dy/ds = m; where l and m are the directional cosines of ‘s’. Therefore dx = l ds and dy = m ds as shown in Fig.8.

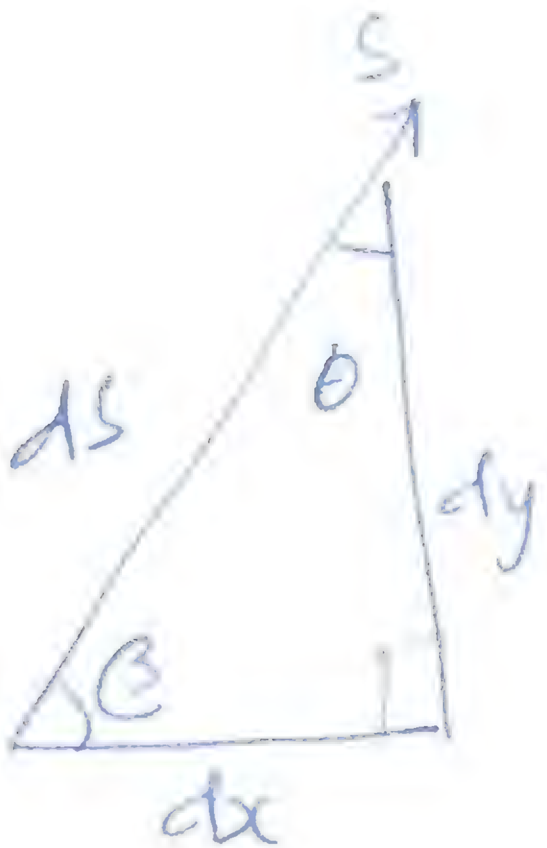


Fig.8

We know from potential line (orthogonal stream line) equation as udx + vdy =0

Substituting in above for ‘u’ and ‘v’ as
$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = l.c$$
$$v = \frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = m.c$$

Therefore, (lc.l + mc.m) ds = 0

($l^2 + m^2$) c ds = 0 , so C.ds = 0 (since $l^2 + m^2 = 1$)

or udx + v dy = cds = 0 ……(3.5).

If this equation is integrated between S_1 and S_2 $\int_{S_1}^{S_2}(u\frac{dx}{ds} + v\frac{dy}{ds})ds = \oint C.ds$,

which is the flow of fluid between S_1 and S_2. It has the dimensions cm^2/sec (L^2T^{-1}). As the velocities u and v are each functions of x and y in the fluid to perform the integration it is necessary to know what these functions are. They can be specified along a given line of flow and the resulting integral along this line is

called a line integral. Similarly a line with angular turns is called a closed line integral.

So the flow around a closed path or the line integral around a closed path is called the circulation 'C' = $\oint u dx + v dy$ ……(3.6)

Thus 'C' is a measure of the extent to which the fluid exhibits rotary motion.

3.2. VORTICITY:

To obtain a quantitative understanding of the rotating fluids with a change of ω, express the motion in Cartesian coordinates of x and y. The rotation is considered for two infinitesimal lines dx and dy in 'x' and 'y' directions. Let 'u' and 'v' are the linear velocity components in y and x directions respectively as shown in Fig.9a. As the angular velocity is different on the two axes

$v = \omega_1 x$ and $-u = -\omega_2 y$ or $\begin{array}{l} \dfrac{\partial v}{\partial x} = \omega_1 \\ -\dfrac{\partial u}{\partial y} = \omega_2 \end{array}$

The average of the two angular velocities say 'ω' is $\omega = (\omega_1+\omega_2)/2 =$

$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\zeta = \omega$ or $\zeta = 2\omega$ ……..(3.7)

where 'ζ' is called the vorticity which is twice the local average angular velocity.

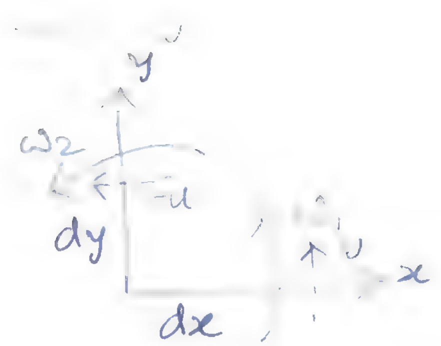


Fig.9a

For cyclonic (anticlockwise in northern hemisphere) flows ζ is positive and for anticyclonic flows (clockwise in N.H), ζ is negative as shown in Fig.9b.

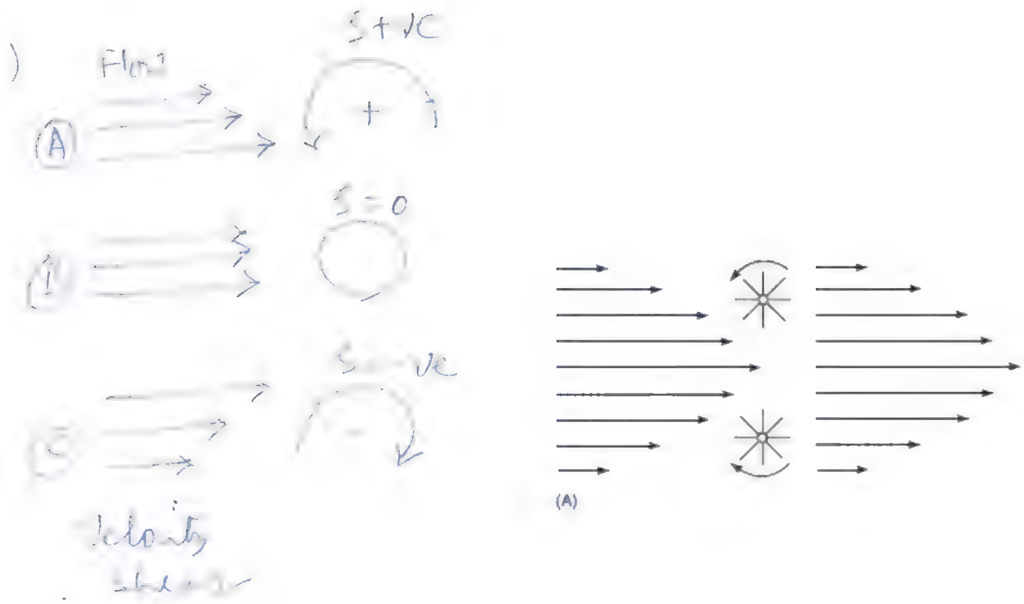


Fig.9b

3.3. PLANETARY VORTICITY:

In the case of earth rotation this is considered as vertical component. We know that the rotation of the earth about its axis results in the deflection of winds by Coriolis force. These deflections are explained in terms of the poleward decrease in the eastward velocity of the surface of the earth. In addition to the linear eastward velocity, the surface of the earth has an angular velocity. Like the linear eastward velocity, this angular velocity depends on latitude. At the poles, the angular velocity, of the surface of the earth is simply Ω (= 7.292×10^{-5} rad sec^{-1}). At lower latitudes, it is a proportion of Ω. The larger the angle between the earth's axis of rotation and the local vertical, the smaller is the angular velocity of the surface of the earth as shown in Fig.10. At the equator, where a vertical axis is at right angles to the axis of rotation of the earth, the angular velocity of the surface is zero. Thus the angular velocity of the surface of the earth about a vertical axis at latitude Φ is given by $\Omega \sin \Phi$. As the vorticity is twice the local average angular velocity (as given by equation 3.7), the planetary vorticity is equal to $2(\Omega \sin \Phi)$ (= f). The planetary vorticity at the poles and the equator respectively are '2Ω' and zero.

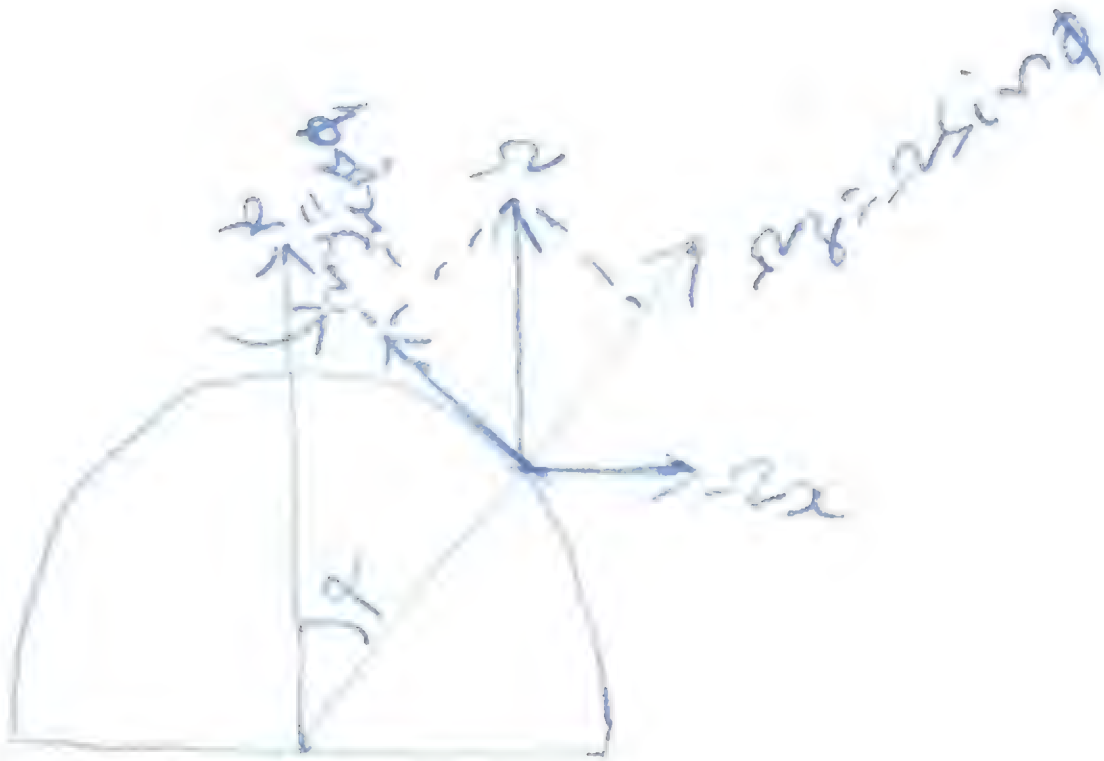


Fig.10.

3.4. ABSOLUTE VORTICITY:

We then define the absolute vorticity as the sum of the two vortices ,($\zeta + f$). The vorticity 'ζ' is measured rotation of the particle relative to the surface of the earth and so it is called relative vorticity.

Considering the frictionless horizontal equation of motion:

$$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial x} + fv$$
$$\frac{dv}{dt} = -\alpha \frac{\partial p}{\partial y} - fu$$

if these are cross differentiated and subtracted one from the other, to eliminate pressure gradient terms, we get: $\frac{d}{dt}(\zeta + f) = -(\zeta + f)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$

Since we know the equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, $\frac{d}{dt}(\zeta + f) = 0$

upto the column of thickness 'D' in the atmosphere called gradient wind level is considered, then $\frac{d}{dt}(\frac{\zeta + f}{D}) = 0$. The term $\frac{\zeta + f}{D}$ is called the potential vorticity.

3.5. VORTICITY THEOREM:

While circulation is a measure of tendency of fluid rotation, vorticity is the same property of fluid for an infinitely small area.

To obtain the vorticity theorem, the horizontal equations of motion are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=fv-\alpha\frac{\partial p}{\partial x}+F_x \ldots\ldots\ldots\ldots\ldots 3.14$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-fu-\alpha\frac{\partial p}{\partial y}+F_y \ldots\ldots\ldots\ldots\ldots 3.15$$

and we know $\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ 3.16

Take partial derivative with respect to (w.r.t) y to equation 3.14 and differentiate w.r.t x to equation 3.15 and subtract 3.14 from 3.15. and re arrange, then we get

$$\frac{\partial}{\partial t}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+u\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+v\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+w\frac{\partial}{\partial z}$$

$$\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+\frac{\partial u}{\partial x}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+\frac{\partial v}{\partial y}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+\left[\frac{\partial v}{\partial z}.\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}.\frac{\partial w}{\partial y}\right]$$

$$=-f\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]-\left[u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}\right]-\alpha[0]-\left[\frac{\partial p}{\partial y}\frac{\partial \alpha}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \alpha}{\partial y}\right]+\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right] \ldots\ldots 3.17$$

$$\frac{\partial}{\partial t}(\zeta)+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+\zeta(D)+\left[\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}\right]=-f.D-v\frac{\partial f}{\partial y}-\left[\frac{\partial p}{\partial y}\frac{\partial \alpha}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \alpha}{\partial y}\right]+\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right]$$

.............(3.18)

since $\frac{\partial f}{\partial x}=0$ and $v\frac{\partial f}{\partial y}=\frac{dy}{dt}.\frac{df}{dy}=\frac{df}{dt}$

It can be further written as

$$\frac{d}{dt}(\zeta)+\frac{d}{dt}(f)=-D(f+\zeta)-\left[\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}\right]-\left[\frac{\partial p}{\partial y}\frac{\partial \alpha}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \alpha}{\partial y}\right]+\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right]$$

$$\frac{d}{dt}(\zeta+f)=-D(f+\zeta)-\left[\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}\right]-\left[\frac{\partial p}{\partial y}\frac{\partial \alpha}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \alpha}{\partial y}\right]+\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right]$$

...........(3.19)

I II III IV

This is called vorticity equation. The term on the left hand side (r.h.s) is called the rate of change of absolute vorticity. This equation says that the absolute vorticity of a parcel of air can change only through the contribution of four terms on the right hand side.

Show that the equation for the time evolution of ζ is

$$\frac{d\zeta}{dt} = -(f+\zeta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)-v\frac{df}{dy}$$

convergence tilting Baroclinic term "beta"-term

….3.19b

3.6. PHYSICAL INTERPRETATION OF THE TERMS OF VORTICITY EQUATION:

The first term (I) on the r.h.s of equation (3.19) is called the divergence (or convergence) term. Positive divergence increases the radius of rotation. As the term is negative, it decreases the magnitude of absolute vorticity.

The second term (II) of equation 3.19 on the r.h.s is known as the tripping (or tilting) term. When the vertical shear ($\frac{\partial w}{\partial x}$) of the first half of this term decreases, horizontal shear ($\frac{\partial v}{\partial z}$) increases upward. As it has minus sign, its contribution to (f+ ζ) is positive. Which means (f+ ζ) will increase with time. A corresponding analogy can easily be made for the second half of this term ($\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}$).

The third term on the r.h.s of equation (3.19) is known as solenoid (or baroclinic) term of the circulation equation (3.13) applied to an infinitely small horizontal circuit.

The last term on the r.h.s of equation 3.19 is the frictional term (or also beta term) that arises due to turbulence in the atmosphere.

Calculation from synoptic data indicates that the contribution of divergence term [(f+ ζ)D] is more than the other terms in large scale weather systems.

So if little divergence exists and other terms are negligible

$$\frac{d}{dt}\left(\frac{\zeta+f}{D}\right)=0 \;\ldots\ldots(3.20)$$

Which means $(f+\zeta)$ is constant. That is in the case of a ring of air parcel revolving round the earth, the absolute vorticity is constant. Which means as 'f' increases, the relative vorticity (ζ) should decrease and vice versa in order that $(f+\zeta)$ to be constant. So if air moves pole ward, 'f' increases and so ζ decreases. Which means pole ward moving air must lose cyclonic curvature and tend toward anti-cyclonic motion. The converse is true for the air moving equator ward. Thus the trajectory of an air parcel in the meridional direction can be drawn using the

constant absolute vorticity (CAV) values plotting on a map as shown in Fig.11. So for weather forecasting CAV trajectories are useful.

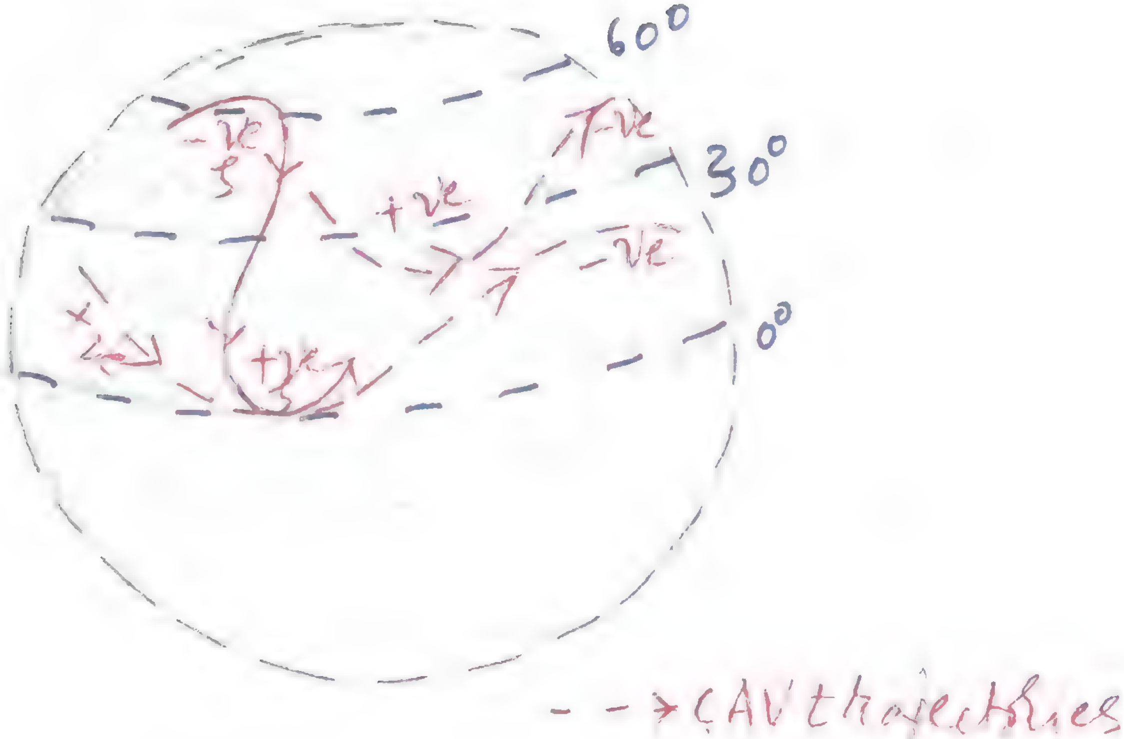


Fig.11. CAV trajectories in the Northern hemisphere

Further if the depth of the column of the air parcel 'D' is considered, the vorticity theorem will be as given in equation 3.20.Where $\left(\frac{\zeta + f}{D}\right)$ is called the potential vorticity which is to be conserved instead of simply absolute vorticity.

Chapter -4
Quasi Geostrophic theory

4.1. POTENTIAL VORTICITY THEOREM:

For the case of incompressible barotropic fluid, the potential vorticity equation can be derived using the continuity equation (2.3):

$$\therefore \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \text{ which means} \therefore \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\partial w}{\partial z} \text{.......(3.23)}$$

The vorticity equation (3.19) can be written with the help of equation (3.23) as

$$\frac{d}{dt}(\zeta + f) = (f + \zeta)\frac{\partial w}{\partial z} \ldots\ldots\ldots\ldots\ldots\ldots(3.24)$$

In the case of barotropic condition, the thermal wind vanishes and so the geostrophic wind is independent of height. Then integrating vertically from $z=D_1$ to D_2 equation (3.24) can be written as

$\frac{d}{dt}(\zeta + f)dz = (f + \zeta)dw$ on integration gives rise to

$$\frac{d}{dt}(\zeta + f)_g [Z]_{D_1}^{D_2} = (f + \zeta)_g W_{wD_1}^{wD_2}$$

$$\frac{d}{dt}(\zeta + f)_g [D_2 - D_1] = (f + \zeta)_g [W_{D2} - W_{D1}] \ldots\ldots\ldots\ldots(3.25)$$

The suffix 'g' denotes under geostrophic condition. But since $W = \frac{dz}{dt}$ we can further write $W_{D2} = \frac{dD_2}{dt}$ and $W_{D1} = \frac{dD_1}{dt}$ and let $H = D_2 - D_1$ is the depth of the column we can write (3.25) as

$$\frac{d}{dt}(\zeta + f)_g [D_2 - D_1] = (f + \zeta)_g \left[\frac{dD_2}{dt} - \frac{dD_1}{dt}\right] = (f + \zeta)\frac{d}{dt}(D_2 - D_1)$$

$H.\frac{d}{dt}(\zeta + f) = (f + \zeta)\frac{d}{dt}(H)$ re arranging the terms

$(\frac{1}{\zeta + f})\frac{d}{dt}(\zeta + f) = \frac{1}{H}\frac{d}{dt}(H)$ which gives rise to

$$\frac{d}{dt}[\ln\left(\frac{\zeta + f}{H}\right)] = 0$$

or $\frac{d}{dt}\left(\frac{\zeta + f}{H}\right) = 0$ This is the potential vorticity equation.

If the flow is purely horizontal (w = o), equation (3.24) reduces to

$$\frac{d}{dt}(\zeta + f) = 0.$$

This is called the barotropic vorticity equation, which states that the absolute vorticity is conserved which means $(\zeta + f)$ = constant.

4.2. THE BASIC EQUATIONS IN DIFFERENT VERTICAL COORDINATES:

Earlier we have seen the equations of motion in Cartesian and spherical polar coordinate systems taking z as vertical coordinate. But modelers find it convenient to use the vertical coordinate as pressure, potential temperature (entropy) or the vertical stability parameter (σ) and derive the equations of momentum, continuity, vorticity etc. As density does not appear in these equations, the complicacy of its variation can easily be eliminated by expressing these equations in a modified vertical coordinate.

4.3. TRANSFORMATION EQUATIONS: ISOBARIC COORDINATE SYSTEM

Since pressure and height are directly related in the atmosphere, it is common to express the variables in terms of pressure surfaces rather than geometrical heights in meteorology. So it is essential to replace the vertical coordinate (z) by pressure (p). Once pressure is taken as vertical coordinate instead of z, it is called isobaric coordinate system.

When 'p' is taken along 'z' direction, the pressure coordinate is not normal to the isobaric surfaces but is exactly vertical upward on the plumb line.

Consider a cross section in x-z plane and let some arbitrary property 'q' is constant along the small section of the surface at an instant of time as shown in Fig.12. Let the variable is $\zeta(x,z,t)$.

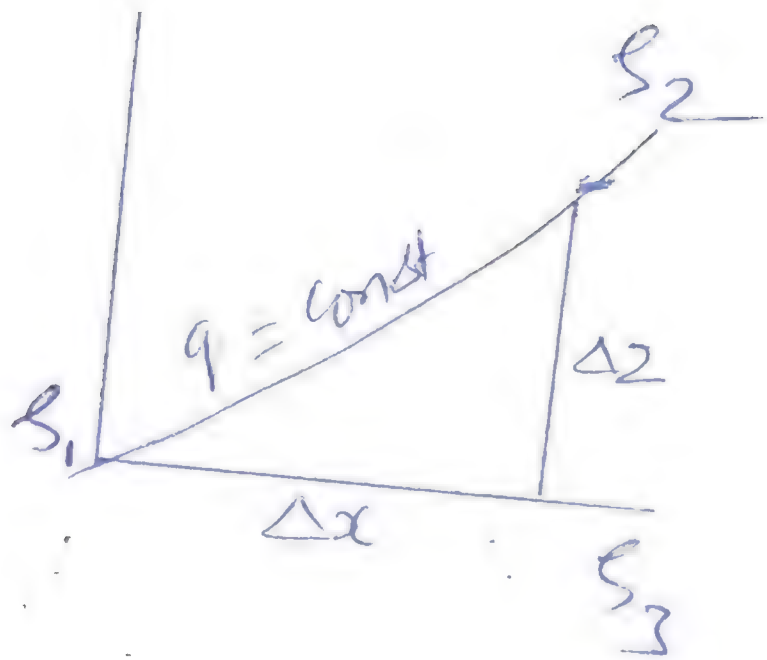


Fig.12

Then we can write

$(\delta\zeta)_{1-2} = (\delta\zeta)_{1-3} + (\delta\zeta)_{2-3}$

for small increments we can write

$\frac{\zeta_2 - \zeta_1}{\Delta x} = \frac{\zeta_3 - \zeta_1}{\Delta x} + \frac{\zeta_2 - \zeta_3}{\Delta z}\frac{\Delta z}{\Delta x}$ which can be written for small increments as

$$\left(\frac{\partial \zeta}{\partial x}\right)_q = \left(\frac{\partial \zeta}{\partial x}\right)_z + \left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right)_q \text{ or } \left(\frac{\partial \zeta}{\partial x}\right)_z = \left(\frac{\partial \zeta}{\partial x}\right)_q - \left(\frac{\partial \zeta}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right)_q \quad \text{..........(4.1)}$$

As we can use any variable in place of ζ we can write (1.1) as

$$\left(\frac{\partial}{\partial x}\right)_z = \left(\frac{\partial}{\partial x}\right)_q - \left(\frac{\partial}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right)_q \quad \text{..........(4.2)}$$

Suffix ‘q’ implies ‘holding q as constant’ in the operation.

For isobaric coordinate system q = p(pressure). A derivative with respect to z can be written as $\frac{\partial}{\partial z} = \frac{\partial p}{\partial z}\frac{\partial}{\partial p}$ and if we make the hydrostatic assumption this becomes $\frac{\partial}{\partial z} = -\rho g \frac{\partial}{\partial p}$ ....(4.3)

So using (4.3), equation (4.2) may be written as

$$\left(\frac{\partial}{\partial x}\right)_z = \left(\frac{\partial}{\partial x}\right)_p + \rho g\left(\frac{\partial z}{\partial x}\right)_p \frac{\partial}{\partial p} \quad \text{..........(4.4)}$$

Similarly we can write for ‘y’ and ‘t’ as

$$\left(\frac{\partial}{\partial y}\right)_z = \left(\frac{\partial}{\partial y}\right)_p + \rho g\left(\frac{\partial z}{\partial y}\right)_p \frac{\partial}{\partial p} \quad \text{..........(4.5)}$$

$$\left(\frac{\partial}{\partial t}\right)_z = \left(\frac{\partial}{\partial t}\right)_p + \rho g\left(\frac{\partial z}{\partial t}\right)_p \frac{\partial}{\partial p} \quad \text{.......... (4.6a)}$$

The last term of equation (4.6a) denotes the rate at which the height of an isobaric surface changes with time. Equations (4.4) to (4.6a) denote the inputs to transform the (x,y,z,t) Cartesian coordinate system to (x,y,p,t) isobaric coordinate system.

4.3.1. ADVANTAGES AND DISADVANTAGES OF ISOBARIC COORDINATE SYSTEM:

The isobaric coordinate system has several advantages for modeling work. Some of them are:

a) meteorological data is normally referred to isobaric surfaces.
b) the continuity equation in this system is reduced and in simple form
c) density does not explicitly appear so no problem of integration or differentiation
d) sound waves are completely filtered.

The serious disadvantage of this system is found in modeling. Modellers use boundary conditions. Here the lower (z_o) boundary condition is assumed coincident with the surface pressure (p_o). we know surface pressure is not constant on the level or geopotential surface. Even when the ground is level also surface pressure changes in the horizontal.

4.3.2.SIGMA COORDINATE SYSTEM:

To avoid the serious disadvantage of the isobaric coordinate system, σ coordinate system has been introduced in modeling work. The disadvantage of isobaric-coordinate system is the coincidence of surface pressure at the lower boundary with the ground (z_0)which is not true.

In the sigma (σ) system, the vertical coordinate is the pressure normalized with the surface pressure (p_s).

$$\therefore \sigma = \frac{p}{p_s} \quad \ldots\ldots\ldots\ldots.(4.6b)$$

This is called the static stability parameter. Thus σ here is a non dimensional independent vertical coordinate which decreases upward from a value $\sigma = 1$ at the ground to $\sigma = 0$ at the top of the atmosphere. Here in σ coordinates the lower boundary condition will always apply exactly at σ =1. Further the vertical sigma velocity is given as

$$\dot{\sigma} = \frac{d\sigma}{dt}$$

This will always be zero at the ground even in the presence of sloping terrain.

So the lower boundary condition in the σ system is $\dot{\sigma} = 0$ at $\sigma = 1$. Thus this σ coordinate system is particularly useful in the regions of strong topographic variations.

4.4. OMEGA EQUATION:

We know the differential variable in x,y,z,t coordinate system as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4.7)$$

For converting this to isobaric coordinate system, apply equation 4.6 to 4.7 then

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_p + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + \rho g\left(\frac{\partial z}{\partial t}\right)_p \frac{\partial}{\partial p}, \quad \ldots\ldots\ldots\ldots\ldots(4.8)$$

[$\because$ comparing equations 4.7 and 4.6, $w\frac{\partial}{\partial z} = \rho g\left(\frac{\partial z}{\partial t}\right)_p \frac{\partial}{\partial p}$ and we know $\frac{\partial z}{\partial t} = w$]

Further equation (4.8) can be written as

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_p + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p - \frac{dp}{dz}\left(\frac{\partial z}{\partial t}\right)_p \frac{\partial}{\partial p}$$

We can write equation (4.9) taking ω(omega)= $-\frac{dp}{dt}$ ……………(4.9)

as

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_p + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p - \frac{dp}{dt}.\frac{\partial}{\partial p} \quad \ldots\ldots\ldots\ldots\ldots(4.10)$$

Here the variable ω(omega)$=-\frac{dp}{dt}$ is equivalent to vertical motion which is like $\frac{dz}{dt}=w$ in x,y,z,t coordinate system. The negative sign implies that the rate of pressure decreases with height. So (4.10) becomes

$$\therefore \frac{d}{dt}=\left(\frac{\partial}{\partial t}\right)_p+u\left(\frac{\partial}{\partial x}\right)_p+v\left(\frac{\partial}{\partial y}\right)_p+\omega.\frac{\partial}{\partial p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.11)$$

This is called the omega equation. Thus the individual derivative in isobaric (x,y,p,t) coordinate system takes the same form as in Cartesian(x,y,z,t) system except that $\omega.\frac{\partial}{\partial p}$ replaces $w\frac{\partial}{\partial z}$. So the variable ω is equivalent to vertical velocity in isobaric coordinate system. When measured in microbars per second, ω is numerically equal to w in cm s^{-1} in the lower atmosphere with opposite sign.

4.5. MOMENTUM EQUATION IN ISOBARIC COORDINATE SYSTEM:

We know in Cartesian (x,y,z,t) coordinate system momentum equations as

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\alpha\frac{\partial p}{\partial x}+fv \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.12)$$

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=--\alpha\frac{\partial p}{\partial y}-fu$$

In a similar way the momentum equations in isobaric (x,y,p,t) coordinate system can be written from equation (4.10) by incorporating u and v as

$$\left(\frac{du}{dt}\right)_p=\left(\frac{\partial u}{\partial t}\right)_p+u\left(\frac{\partial u}{\partial x}\right)_p+v\left(\frac{\partial u}{\partial y}\right)_p+\omega.\frac{\partial u}{\partial p}=-g\frac{\partial z}{\partial x}+fv \quad \ldots\ldots\ldots(4.12a)$$

$$\left(\frac{dv}{dt}\right)_p=\left(\frac{\partial v}{\partial t}\right)_p+u\left(\frac{\partial v}{\partial x}\right)_p+v\left(\frac{\partial v}{\partial y}\right)_p+\omega.\frac{\partial v}{\partial p}=-g\frac{\partial z}{\partial y}-fu \quad \ldots\ldots\ldots\ldots(4.12b)$$

Please note $\alpha\partial p=g\partial z$ is used here to the first term of the right hand side. This kind of momentum equations are used by modelers because the density term does not appear explicitly.

4.7. THE VORTICITY EQUATION IN ISOBARIC COORDINATE SYSTEM:

This form of vorticity equation can be obtained by differentiating momentum equation in isobaric coordinates (4.12a) with respect to(w.r.t) 'y' and the equation (4.12b) with respect to 'x' and subtract the former from the latter holding 'p' as constant. Then we get

$$\left(\frac{\partial u}{\partial t}\right)_p+u\left(\frac{\partial u}{\partial x}\right)_p+v\left(\frac{\partial u}{\partial y}\right)_p+\omega.\frac{\partial u}{\partial p}=-g\left(\frac{\partial z}{\partial x}\right)_p+fv \quad \ldots\ldots\ldots(4.12a)$$

$$\left(\frac{\partial v}{\partial t}\right)_p + u\left(\frac{\partial v}{\partial x}\right)_p + v\left(\frac{\partial v}{\partial y}\right)_p + \omega.\frac{\partial v}{\partial p} = -g\left(\frac{\partial z}{\partial y}\right)_p - fu \quad \text{...........(4.12b)}$$

Differentiating (4.12a) w.r.t 'y' we get

$$\left(\frac{\partial^2 u}{\partial y \partial t}\right) + u\left(\frac{\partial^2 u}{\partial x \partial y}\right) + \frac{\partial u}{\partial x}.\frac{\partial u}{\partial y} + v\left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial y} + \omega.\frac{\partial^2 u}{\partial y \partial p} + \frac{\partial u}{\partial p}.\frac{\partial \omega}{\partial y} = -g\frac{\partial^2 z}{\partial x \partial y} + f\frac{\partial v}{\partial y} + \frac{\partial f}{\partial y}v \quad \text{.........(4.19)}$$

Differentiating (4.12b) w.r.t 'x' we get

$$\left(\frac{\partial^2 v}{\partial x \partial t}\right) + u\left(\frac{\partial^2 v}{\partial x^2}\right) + \frac{\partial v}{\partial x}.\frac{\partial u}{\partial x} + v\left(\frac{\partial^2 v}{\partial x \partial y}\right) + \frac{\partial v}{\partial y}.\frac{\partial v}{\partial x} + \omega.\frac{\partial^2 v}{\partial x \partial p} + \frac{\partial v}{\partial p}.\frac{\partial \omega}{\partial x} = -g\frac{\partial^2 z}{\partial x \partial y} - f\frac{\partial u}{\partial x} - u\frac{\partial f}{\partial x} \quad \text{.........(4.20)}$$

Subtracting (4.19) from (4.20) and taking $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \zeta$ = k .(∇ .V) (we get

$$\left(\frac{\partial \zeta}{\partial t}\right) + u\left(\frac{\partial \zeta}{\partial x}\right) + v\left(\frac{\partial \zeta}{\partial y}\right) + \omega.\frac{\partial \zeta}{\partial p} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial p}.\frac{\partial \omega}{\partial x} - \frac{\partial u}{\partial p}.\frac{\partial \omega}{\partial y} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - u\frac{\partial f}{\partial x} - v\frac{\partial f}{\partial y}$$

$$\left(\frac{\partial \zeta}{\partial t}\right)_p + u\left(\frac{\partial \zeta}{\partial x}\right)_p + v\left(\frac{\partial \zeta}{\partial y}\right)_p + \omega.\frac{\partial \zeta}{\partial p} = (-f + \zeta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p - v\beta - \left(\frac{\partial \omega}{\partial x}\right)_p \frac{\partial v}{\partial p}. + \left(\frac{\partial \omega}{\partial y}\right)_p \frac{\partial u}{\partial p}. \quad \text{.........(4.21)}$$

This is the vorticity equation in isobaric coordinate system. The first term on the r.h.s is divergence term, the second term is β plane term and the last two terms considering as the third term is tripping term. Note that the solenoidal term is missing here. This is because the isobaric system doesn't depend on the density varaiation.

This can be further written taking $\frac{\partial f}{\partial x} = 0$ as the coriolis parameter (f) at any latitude is constant and $\frac{\partial f}{\partial y} = \beta$, variation of coriolis parameter with latitude.

$$\frac{\partial \zeta}{\partial t} = -V.\nabla(\zeta + f) - \omega.\frac{\partial \zeta}{\partial p} - (\zeta + f)\nabla.V - \frac{\partial v}{\partial p}.\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial p}.\frac{\partial \omega}{\partial y} \quad \text{where V = iu +jv ......(4.22)}$$

4.8. QUASI-GEOSTROPHIC VORTICITY EQUATION:

Referring to the vorticity equation in isobaric coordinate system in equation (4.22), the terms reading from left to right are as follows:

1. The local rate of change of relative vorticity
2. the horizontal advection of absolute vorticity
3. the vertical advection of relative vorticity

4. the divergence term
5. the twisting or tilting or tripping term

As shown by the scale analysis earlier we simplify the vorticity equation for midlatitude synoptic scale motions by

1. neglecting the vertical advection and twisting terms
2. neglecting ζ compared to f in the divergence terms
3. approximating the horizontal velocity by the geostrophic wind in the advecting term and
4. replacing the relative vorticity by its geostrophic value and
5. expressing coriolis parameter in Taylor's series with initial value f_o at latitude φ_o such that

$$f = f_o + \beta y + \text{higher order terms} \quad \ldots\ldots\ldots\ldots(4.23)$$

where $\beta = \left(\frac{df}{dy}\right)_{\phi_o}$ and y = 0 at φ_o. Neglect the higher order terms.

β plane approximation:

Let L designate the latitudinal scale of motion, then the ratio of the first two terms of 'f' has the order of magnitude as

$$\frac{\beta L}{f_0} = \frac{\frac{\partial f_0}{\partial y} L}{f_0} = \frac{d(2\Omega \sin\phi_0)L}{2\Omega \sin\phi_0 . a} = \frac{(2\Omega \cos\phi_0)L}{2\Omega \sin\phi_0 . a}$$

Thus when the latitudinal scale (L) of motions is small compared to the radius of the earth (a), (then L<a) we can let the Coriolis parameter f_0 has a constant value except where it appears differentiated in the advection term. In such case $\frac{df}{dy}$ = β is assumed to be constant. This approximation is usually referred to as the beta-palne approximation.

Applying the above assumptions and approximations, equation (4.22) becomes:

$$\frac{\partial \zeta_g}{\partial t} = -V_g . \nabla(\zeta_g + f) - (f_0)\nabla . V \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.24)$$

Where $\zeta_g = \frac{\nabla^2 \varphi}{f_0}$ and $V_g = k \times \frac{\nabla \varphi}{f_0}$

This is called quasi-geostrophic vorticity equation .

It is important to note from equation (4.24), in the divergence term horizontal wind is not replaced by its geostrophic value. This is because the small departures of the horizontal wind from geostrophy accounts for the divergence.

We know the continuity equation from equation (4.17)

$$\left(\frac{\partial u}{\partial x}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p = \nabla . V = - \frac{\partial \omega}{\partial p} \quad \ldots\ldots\ldots(4.25)$$

Substituting this equation (4.25) in (4.24) we get

$$\frac{\partial \zeta_g}{\partial t} = -V_g.\nabla(\zeta_g + f) + (f_0)\frac{\partial \omega}{\partial p} \quad \ldots\ldots\ldots\ldots\ldots(4.26)$$

This is the alternative form of quasi geostrophic vorticity equation. The advantage of this equation is ω estimates are better than the estimates based on continuity equation (4.17) because both the local change of geostrophic vorticity and the advection of vorticity can be estimated accurately as they are related to geopotential. This equation (4.26) forms as a closed prediction equation in ω .